Indian Statistical Institute, Bangalore

M. Math First Year

First Semester - Measure Theoretic Probability

Final Exam | Date: November 11, 2019

Maximum marks: 50 | Duration: 3 hours

Section I: Answer any four and each question carries 6 Marks

1. (a) Prove that a set of outer measure zero is Lebesgue measurable (*Marks: 3*).

   (b) Prove that an union of two Lebesgue measurable sets is Lebesgue measurable.

2. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and (f_n) be a sequence of measurable functions converging everywhere on X. Prove that for each $\epsilon > 0$ there is a $A \in \mathcal{A}$ such that $\mu(A) < \epsilon$ and (f_n) converges uniformly on $X \setminus A$.

3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function such that f_x is Borel measurable and f_y is continuous. Prove that f is measurable.

4. Prove that a sequence (X_n) of random variables converge a.e. to 0 if and only if $P(|X_n| > \epsilon \text{ i.o. }) = 0$ for all $\epsilon > 0$.

5. Let X be a random variable. Is there a m such that $P(X \leq m) \geq \frac{1}{2}$ and $P(X \geq m) \geq \frac{1}{2}$? Is it unique? Justify your answer.

6. Prove the Levy-Cramer continuity Theorem.

Section II: Answer any two and each question carries 13 Marks

1. (a) If μ is a Borel measure on $\mathbb{R}$ such that $\mu([0,1]) = 1$ and $\mu(x + B) = \mu(B)$ for all $x \in X$, all Borel sets B in $\mathbb{R}$, prove that μ is the Lebesgue measure.

   (b) Let μ and ν be two σ-finite measures on X. Prove that $\nu = \nu_0 + \nu_1$ such that $\nu_0 \ll \mu$ and $\nu_1 \perp \mu$ (*Marks: 7*).

2. (a) Let X, X_n be random variables. Prove that $X_n \to X$ almost everywhere if and only if $\lim_m P(|X_n - X| \leq \epsilon \text{ for all } n \geq m) = 1$ for any $\epsilon > 0$ (*Marks: 7*).

   (b) For any sequence (X_n) of random variables, $X_n \to +\infty$ if and only if $P(X_n < M \text{ i.o. }) = 0$ for any $M > 0$.

3. (a) A sequence of probability measures (μ_n) converges weakly to a probability measure μ if and only if $\overline{\lim}\mu_n(C) \leq \mu(C)$ for any closed set C of $\mathbb{R}$ (*Marks: 7*).

   (b) If μ and λ are probability measures on $\mathbb{R}$ such that $\mu * \lambda = \mu$, find λ.